1) Maxwell equations

1. Intro: Maxwell equations

EE-624, 3 credits, 3-5 individual homeworks

Tuesdays, 9:15 - 11 am, ELG 120

A broad set of key concepts to understand and do research in electromagnetism / photonics / wave physics

I) Continuous media

II) Scattering

III) Photonic crystals

IV) Metamaterials

Maxwell's equations

$$\begin{cases} \vec{\nabla} \cdot \vec{\mathcal{D}}(\vec{r},t) = \rho(\vec{r},t) & \text{Maxwell - Gauss} \\ \vec{\nabla} \cdot \vec{\mathcal{B}}(\vec{r},t) = \rho_m(\vec{r},t) & \\ \vec{\nabla} \times \vec{\mathcal{E}}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{r},t) - \vec{\mathcal{J}}_m(\vec{r},t) & \text{Maxwell - Faraday} \\ \vec{\nabla} \times \vec{\mathcal{H}}(\vec{r},t) = \frac{\partial}{\partial t}\vec{\mathcal{D}}(\vec{r},t) + \vec{\mathcal{J}}(\vec{r},t) & \text{Maxwell - Ampère} \end{cases}$$

$\vec{\mathcal{E}}$ electric field [V/m]
$\vec{\mathcal{H}}$ magnetic field [A/m]
$\vec{\mathcal{D}}$ elec. flux density [C/m²]
$\vec{\mathcal{B}}$ mag. flux density [Wb/m²]

fields

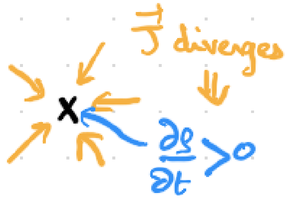
ρ electric charge density [C/m³]
ρ_m magnetic charge density [Wb/m³] = 0
$\vec{\mathcal{J}}$ electric current density [A/m²]
$\vec{\mathcal{J}}_m$ mag. current density [Wb/m²s] = 0

sources

2. Charge conservation

Maxwell - Faraday : $\vec{\nabla} \cdot \vec{\nabla} \times \vec{\mathcal{H}} = \frac{\partial}{\partial t} \overbrace{\vec{\nabla} \cdot \vec{\mathcal{D}}}^{\rho} + \vec{\nabla} \cdot \vec{J} = 0 \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$

continuity equation



charge is conserved

3. Constitutive relations

* Constitutive relations $\vec{\mathcal{D}} = f(\vec{\mathcal{E}}, \vec{\mathcal{H}})$ $\quad \vec{\mathcal{B}} = g(\vec{\mathcal{E}}, \vec{\mathcal{H}})$ describe the material with f, g continuous functions within a continuous material domain.

* assumptions :

  - linearity : f, g are linear for small field amplitude $\neq$ non-linear
  - space-time homogeneity : $f(\cancel{\vec{r}}, \cancel{t}, \mathcal{E}(\vec{r},t), \vec{\mathcal{H}}(\vec{r},t))$ $\neq$ inhomogeneous, time-varying
  - locality : f, g functions of $\mathcal{E}$ and $\mathcal{H}$ at the same point ie :

$$\vec{\mathcal{D}}(\vec{r}) = f(\vec{\mathcal{E}}(\vec{r}), \vec{\mathcal{H}}(\vec{r}))$$ $\neq$ non-local = spatially dispersive

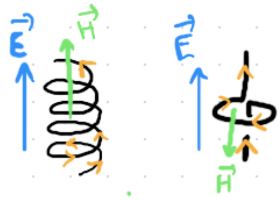
  - non (time) dispersive : $\vec{\mathcal{D}}(t) = f(\vec{\mathcal{E}}(t), \vec{\mathcal{H}}(t))$ $\neq$ dispersive

    if time-dispersive, f & g must satisfy causality

  - Chirality : $\dfrac{\partial \mathcal{D}_i}{\partial \mathcal{H}_j} \neq 0$, $\dfrac{\partial \mathcal{B}_k}{\partial \mathcal{E}_l} \neq 0$ [magneto-electric coupling]

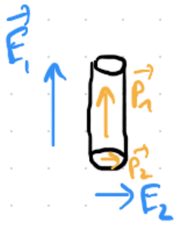
  - Anisotropic : $\vec{\mathcal{D}}$ and $\vec{\mathcal{E}}$ have $\neq$ directions

CHIRAL
$$\begin{cases} \vec{\mathcal{D}} = \varepsilon \vec{\mathcal{E}} + \chi_1 \vec{\mathcal{H}} \\ \vec{\mathcal{B}} = \mu \vec{\mathcal{H}} + \chi_2 \vec{\mathcal{E}} \end{cases}$$



with random orientations

ANISOTROPIC
$$\begin{cases} \vec{\mathcal{D}} = \hat{\varepsilon} \vec{\mathcal{E}} \\ \vec{\mathcal{B}} = \hat{\mu} \vec{\mathcal{H}} \end{cases}$$



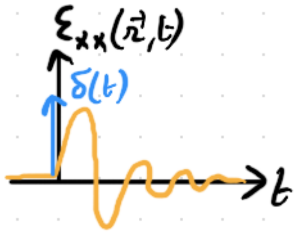
periodic $\Rightarrow$ $\hat{\varepsilon} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_2 \end{pmatrix}$

BI-ANISOTROPIC
$$\begin{cases} \vec{\mathcal{D}} = \hat{\varepsilon} \vec{\mathcal{E}} + \hat{\alpha} \vec{\mathcal{H}} \\ \vec{\mathcal{B}} = \hat{\mu} \vec{\mathcal{H}} + \hat{\beta} \vec{\mathcal{E}} \end{cases}$$

* General non-local, time dependent, and dispersive (in space & time) linear model :

$$\vec{\mathcal{D}}(\vec{r},t) = \int_0^\infty dt' \int_{\infty^3} d^3\vec{r}' \left[\hat{\varepsilon}(\vec{r}-\vec{r}', t-t')\,\vec{\mathcal{E}}(\vec{r}',t') + \hat{\alpha}(\vec{r}-\vec{r}', t-t')\,\vec{\mathcal{H}}(\vec{r}',t')\right]$$

$$\vec{\mathcal{B}}(\vec{r},t) = \int_0^\infty dt' \int_{\infty^3} d^3\vec{r}' \left(\hat{\mu}(\vec{r}-\vec{r}', t-t')\,\vec{\mathcal{H}}(\vec{r}',t') + \hat{\beta}(\vec{r}-\vec{r}', t-t')\,\vec{\mathcal{E}}(\vec{r}',t')\right]$$



convolution = impulse response

$\Rightarrow \delta$ function if ideal

local — no inertia/delay

simplification 1 $\hat{\varepsilon}(\vec{r}-\vec{r}', t-t') = \hat{\varepsilon}_t\,\delta(\vec{r}-\vec{r}')\,\delta(t-t')$ (local non-dispersive)

$$\Rightarrow \begin{cases} \vec{\mathcal{D}}(\vec{r},t) = \hat{\varepsilon}_t\,\vec{\mathcal{E}}(\vec{r},t) + \hat{\alpha}_t\,\vec{\mathcal{H}}(\vec{r},t) \\ \vec{\mathcal{B}}(\vec{r},t) = \hat{\mu}_t\,\vec{\mathcal{H}}(\vec{r},t) + \hat{\beta}_t\,\vec{\mathcal{E}}(\vec{r},t) \end{cases}$$

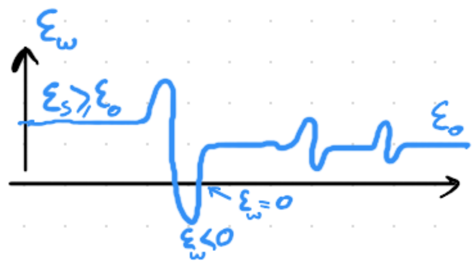
simplification 2 : $\hat{\varepsilon}(\vec{r}-\vec{r}', t-t') = \hat{\varepsilon}_t(t-t')\,\delta(\vec{r}-\vec{r}')$ (local only)

Phasors : $\vec{\mathcal{E}}(\vec{r},t) = \mathrm{Re}\left[\vec{E}(\vec{r})\,e^{j\omega t}\right]$

↳ phasor

$$\begin{cases} \vec{D}(\vec{r}) = \hat{\varepsilon}_\omega\,\vec{E}(\vec{r}) + \hat{\alpha}_\omega\,\vec{H}(\vec{r}) \\ \vec{B}(\vec{r}) = \hat{\mu}_\omega\,\vec{H}(\vec{r}) + \hat{\beta}_\omega\,\vec{E}(\vec{r}) \end{cases}$$

↑ Fourier transforms ↑



∃ frequency ranges where we can neglect time dispersion (ex: dielectrics at optical freq)

4. Boundary conditions

on S
$$
\begin{cases}
\vec{n}\cdot(\vec{D}_2-\vec{D}_1) = \rho_s \overset{\text{often}}{=} 0 \\
\vec{n}\cdot(\vec{B}_2-\vec{B}_1) = \rho_{ms} = 0 \\
\vec{n}\times(\vec{E}_2-\vec{E}_1) = -\vec{J}_{ms} = \vec{0} \\
\vec{n}\times(\vec{H}_2-\vec{H}_1) = \vec{J}_s \overset{\text{often}}{=} \vec{0}
\end{cases}
$$